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ON THE LOCATION OF FIXED POINTS ON PAIRS OF SPACES*

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Let $f:(X, A)\to(X, A)$ be an admissible selfmap of a pair of metrizable ANR's. A Nielsen number of the complement $\tilde{N}(f; X, A)$ and a Nielsen number of the boundary $\tilde{n}(f; X, A)$ are defined. $\tilde{N}(f; X, A)$ is a lower bound for the number of fixed points on $\mathrm{Cl}(X-A)$ for all maps in the homotopy class of f. It is usually possible to homotope f to a map which is fixed point free on Bd A, but maps in the homotopy class of f which have a minimal fixed point set on X must have at least $\tilde{n}(f; X, A)$ fixed points on Bd A. It is shown that for many pairs of compact polyhedra these lower bounds are the best possible ones, as there exists a map homotopic to f with a minimal fixed point set on X which has exactly $\tilde{N}(f; X, A)$ fixed points on $\mathrm{Cl}(X-A)$ and $\tilde{n}(f; X, A)$ fixed points on Bd A. These results, which make the location of fixed points on pairs of spaces more precise, sharpen previous ones which show that the relative Nielsen number $N(f; X, A)$ is the minimum number of fixed points on all of X for selfmaps of (X, A), as well as results which use Lefschetz fixed point theory to find sufficient conditions for the existence of one fixed point on $\mathrm{Cl}(X-A)$.

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1. Introduction

Let $f:(X, A)\to(X, A)$ be a selfmap of a pair of spaces. Minimal fixed point sets of f have been studied in [7], where a relative Nielsen number $N(f; X, A)$ was introduced, and where it was shown that $N(f; X, A)$ is a lower bound for the number of fixed points on X for all maps in the homotopy class of f if (X, A) is a pair of compact metrizable ANR's. For many pairs of compact polyhedra $N(f; X, A)$ is actually the best possible lower bound, as there exists a map $g:(X, A)\to(X, A)$ homotopic to f which has precisely $N(f; X, A)$ fixed points [7, Theorem 6.2]. But

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while the position of minimal fixed point sets on X is usually arbitrary if $A=\emptyset$ (and hence $N(f; X, A)=N(f)$), this is no longer true if $A\neq\emptyset$. This fact is the topic of the present paper.

In the special case where f is a deformation (i.e. homotopic to the identity map $\mathrm{id}:(X, A)\to(X, A)$) the possible location of minimal and arbitrary fixed point sets was described in terms of the Euler characteristics of X and the components of A in [8]. We now extend some of these results to arbitrary maps of pairs, and study the number of fixed points which must occur on the closure $\mathrm{Cl}(X-A)$ of the complement of A, on the boundary $\mathrm{Bd}\,A$ of A, and on A. Thus the location of fixed point sets for selfmaps of pairs of spaces is made more precise.

The topic of this paper dates back to 1968, when Bowszyc [1] established sufficient conditions for the existence of at least one fixed point on $\mathrm{Cl}(X-A)$ in terms of Lefschetz numbers for compact selfmaps of pairs of metrizable ANR's. His work has been extended to maps of compact attraction of such pairs by Dzedzej [3] and Górniewicz and Granas [4]. (See e.g. Theorem 4.3 below.) By using Nielsen fixed point theory rather than Lefschetz fixed point theory we can obtain much stronger results.

Our methods are related to those of [7]. We introduce in Section 3 two variants of the relative Nielsen number which we call the Nielsen number of the complement and the Nielsen number of the boundary, and denote by $\tilde{N}(f; X, A)$ and $\tilde{n}(f; X, A)$. They are defined for a class of selfmaps of pairs of metrizable ANR's which are not necessarily compact. This class is modelled on the class of "admissible" selfmaps of metrizable ANR's which was used by Scholz [9] in order to define the Nielsen number $N(f)$ for maps $f: X\to X$ of a non-compact metrizable ANR X. Admissible selfmaps of pairs of metrizable ANR's are defined in Section 2, and the relative Nielsen number $N(f; X, A)$ of [7] is extended to this setting.

In Section 3 we prove that $\tilde{N}(f; X, A)$ is a lower bound for the number of fixed points on $\mathrm{Cl}(X-A)$ for admissible selfmaps of the pair (X, A) (Theorem 3.1). But $\tilde{n}(f; X, A)$ is not always a lower bound for the number of fixed points on $\mathrm{Bd}\,A$, as a selfmap of (X, A) can usually be homotoped to one which is fixed point free on $\mathrm{Bd}\,A$ (Theorem 3.2). Nevertheless $\tilde{n}(f; X, A)$ plays the role of a lower bound, as any selfmap of (X, A) with a *minimal* fixed point set must have at least $\tilde{n}(f; X, A)$ fixed points on $\mathrm{Bd}\,A$ (Theorem 3.5). We also show how the Nielsen numbers of the complement and of the boundary can be calculated in some special cases and in concrete examples. In Section 4 we show that $\tilde{N}(f; X, A)$ and $\tilde{n}(f; X, A)$ have the usual basic properties of Nielsen numbers, especially that they are homotopy invariant (Theorem 4.1), and we relate $\tilde{N}(f; X, A)$ to the results of Bowszyc (Theorem 4.4). The final Section 5 gives conditions for a pair of compact polyhedra (X, A) which ensure that $\tilde{N}(f; X, A)$ and $\tilde{n}(f; X, A)$ are optimal lower bounds for maps with minimal fixed point sets, i.e. conditions which allow the construction of a map $g:(X, A)\to(X, A)$ homotopic to a given map $f:(X, A)\to(X, A)$ so that g has $N(f; X, A)$ fixed points of which $\tilde{N}(f; X, A)$ lie on $\mathrm{Cl}(X-A)$ and $\tilde{n}(f; X, A)$ lie on $\mathrm{Bd}\,A$ (Theorem 5.1). The conditions on (X, A) arise from the fact that the proof of Theorem 5.1 consists of sharpening some methods used in the proof of the Minimum Theorem 6.2 of [7].

The paper requires knowledge of some results and proofs from [7] and [9]. Background material on Nielsen fixed point theory can be found in [2] and [6].

2. Relative Nielsen numbers for pairs of non-compact ANR's

We give in this section the necessary background concerning fixed point classes and the relative Nielsen number $N(f; X, A)$ for a map $f:(X, A)\to(X, A)$ of pairs of spaces. These concepts were introduced in [7] for the case where (X, A) is a pair of compact metrizable ANR's, and are here presented in a more general setting which reduces for $A=\emptyset$ to that of Scholz [9].

Let $f:(X, A)\to(X, A)$ be a map of a pair of spaces. We write $\bar{f}:A\to A$ for the restriction of f to A, and $f:X\to X$ if the condition $f(A)\subset A$ is immaterial. Hence homotopies of $f:(X, A)\to(X, A)$ are maps of the form $H:(X\times I, A\times I)\to(X, A)$, and homotopies of $f:X\to X$ are maps of the form $H:X\times I\to X$. We write $\mathrm{Fix}\, f = \{x\in X \mid f(x)=x\}$ for the fixed point set of f, and $\mathrm{ind}(X, f, \mathbb{F})$ for the index of the fixed point class $\mathbb{F}$ of $f: X\to X$. The closure, interior and boundary in X of a subspace $B\subset X$ are denoted by $\mathrm{Cl}\, B$, $\mathrm{Int}\, B$ and $\mathrm{Bd}\, B$. Other notation is as in [9].

A class $\mathcal{F}$ of selfmaps $f:(X, A)\to(X, A)$ of pairs of spaces (X, A) will be called *admissible* if, for each $f\in\mathcal{F}$, the maps $f:X\to X$ and $\bar{f}; A\to A$ induced by $f:(X, A)\to(X, A)$ are admissible in the sense of [9, p. 82], i.e. if

(i) $f:X\to X$ and $\bar{f}:A\to A$ have generalized Lefschetz numbers $L(f)$ and $L(\bar{f})$,

(ii) $\mathrm{Fix}\, f$ is compact in X and $\mathrm{Fix}\, \bar{f}$ is compact in A,

(iii) X and A are metrizable ANR's.

An $\mathcal{F}$-homotopy is a map $H:(X\times I, A\times I)\to(X, A)$ so that both $H:X\times I\to X$ and $\bar{H}:A\times I\to A$ are $\mathcal{F}$-homotopies as defined in [9, p. 81]. The class $\mathcal{F}$ *admits an index* if, for each $f\in\mathcal{F}$, an index which satisfies the five axioms of [9, p. 82] exists for all maps $f:X\to X$ and $\bar{f}:A\to A$ induced by $f\in\mathcal{F}$. We shall, in the remainder of this paper, always assume that $\mathcal{F}$ is an admissible class of selfmaps of pairs of spaces which admits an index. Examples of such classes include the class of compact selfmaps of pairs of metrizable ANR's (X, A), where A is closed in X, considered by Bowszyc [1] and the more general class of maps of compact attraction of pairs of metrizable ANR's considered in [3] and [4].

For any $f\in\mathcal{F}$ the fixed point classes $\mathbb{F}$ of $f:X\to X$ and $\bar{\mathbb{F}}$ of $\bar{f}:A\to A$ are defined as in [9]. Hence fixed point classes in this paper are always considered to be non-empty. As in [7, Definition 2.1 and Corollary 2.3] we say that a fixed point class $\mathbb{F}$ of $f:X\to X$ is a *common fixed point class of f and $\bar{f}$* if $\mathbb{F}$ contains at least one essential fixed point class $\bar{\mathbb{F}}$ of $\bar{f}:A\to A$. An *essential common fixed point class of f and $\bar{f}$* is an essential fixed point class of $f:X\to X$ which is a common fixed point class of f and $\bar{f}$. We write $N(f, \bar{f})$ for the number of essential common fixed point classes of f and $\bar{f}$ and define the *relative Nielsen number $N(f; X, A)$* of the map $f \in \mathcal{F}$ as

$$N(f; X, A) = N(\bar{f}) + N(f) - N(f, \bar{f}). \tag{2.1}$$

As the ordinary Nielsen numbers $N(f)$ and $N(\bar{f})$ are invariant under $\mathcal{F}$-homotopies, the invariance of $N(f; X, A)$ under $\mathcal{F}$-homotopies can be proved as in [7, Theorem 3.3]. All other results from [7, Section 2 and 3] hold for maps in $\mathcal{F}$. In particular one can use the same counting argument as in the proof of [7, Theorem 3.1] to obtain the following theorem.

Theorem 2.2. *If $f:(X, A) \to (X, A)$ is a map in $\mathcal{F}$, then any map $\mathcal{F}$-homotopic to f has at least $N(f; X, A)$ fixed points.*

3. The Nielsen numbers of the complement and of the boundary

We now introduce two numbers which are related to the numbers of Section 2. The first, which we call the Nielsen number of the complement, will be a lower bound for the number of fixed points on $\mathrm{Cl}(X - A)$, and the second, which we call the Nielsen number of the boundary, will be a lower bound for the number of fixed points on $\mathrm{Bd}\, A$ for selfmaps of (X, A) with minimal fixed point sets. If $f \in \mathcal{F}$ and if $\mathbb{F}$ is a fixed point class of $f: X \to X$, then $\mathbb{F} \cap A$ is empty or the union of fixed point classes of $\bar{f}: A \to A$ [7, Lemma 2.2] and hence the index $\mathrm{ind}(A, \bar{f}, \mathbb{F} \cap A)$ is well defined as the sum of the indices of the fixed point classes of $\bar{f}: A \to A$ which are contained in $\mathbb{F} \cap A$. We say that the fixed point class $\mathbb{F}$ of $f: X \to X$ *assumes its index in A* if

$$\mathrm{ind}(X, f, \mathbb{F}) = \mathrm{ind}(A, \bar{f}, \mathbb{F} \cap A).$$

The Nielsen number of the complement $\tilde{N}(f; X, A)$ is defined as the number of all fixed point classes of $f: X \to X$ which do not assume their index in A, and the *Nielsen number of the boundary* $\tilde{n}(f; X, A)$ is defined as the number of fixed point classes of $f: X \to X$ which do not assume their index in A and are common fixed point classes of f and $\bar{f}$. Hence $\tilde{n}(f; X, A) \leq \tilde{N}(f; X, A)$ and both are non-negative integers. We want to stress that in the calculation of $\tilde{N}(f; X, A)$ and $\tilde{n}(f; X, A)$ *all* (non-empty) fixed point classes of $f: X \to X$, whether essential or inessential, have to be considered. This may seem strange at first, but becomes more natural once one realizes that an inessential fixed point class of $f: X \to X$ which does not assume its index in A must contain at least one essential fixed point class of $\bar{f}: A \to A$. The tilde over N and n in the symbols $\tilde{N}(f; X, A)$ and $\tilde{n}(f; X, A)$ was chosen to remind the reader that they cannot be computed (as in the case of $N(f)$ and $N(f; X, A)$) from the knowledge of essential fixed point classes only. The next theorem explains the name of $\tilde{N}(f; X, A)$.

Theorem 3.1 (Lower bound). *If $f:(X, A) \to (X, A)$ is a map in $\mathscr{F}$, then f has at least $\tilde{N}(f; X, A)$ fixed points on* $\mathrm{Cl}(X-A)$.

Proof. Let $\mathbb{F}$ be a fixed point class of $f: X \to X$ which does not assume its index in A. Then $\mathbb{F} \cap \mathrm{Cl}(X-A) \neq \emptyset$, for otherwise $\mathbb{F} \subset \mathrm{Int}\, A$ would imply

$$\mathrm{ind}(X, f, \mathbb{F}) = \mathrm{ind}(A, \bar{f}, \mathbb{F}) = \mathrm{ind}(A, \bar{f}, \mathbb{F} \cap A).$$

Therefore we can find $\tilde{N}(f; X, A)$ fixed points on $\mathrm{Cl}(X-A)$, one for each such fixed point class $\mathbb{F}$. $\square$

In contrast to Theorem 3.1 it is not true that $\tilde{n}(f; X, A)$ is always a lower bound for the number of fixed points on Bd A. In fact it is usually possible to homotope f to a selfmap g of (X, A) which is fixed point free on Bd A, as the proof of [7, Theorem 4.1] can easily be adapted so that the $N(\bar{f})$ fixed points of $\bar{g}: A \to A$ are moved to Int A rather than Bd A. Recall from [7, Section 4] that a space X is called a *Nielsen space* if every map $f: X \to X$ is homotopic to a map $g: X \to X$ which has $N(f)$ fixed points, and if these fixed points can lie anywhere on X. Thus we have the following theorem.

Theorem 3.2. *Let (X, A) be a pair of compact polyhedra, let X be connected and let each component of A be a Nielsen space. Then every map $f:(X, A) \to (X, A)$ is homotopic to a map $g:(X, A) \to (X, A)$ which has no fixed points on* Bd A.

Here is a concrete example of a map with $0 < \tilde{n}(f; X, A)$ but without fixed points on Bd A.

Example 3.3. Let $X = S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ be the unit circle in the plane $\mathbb{R}^2$, let $a_0 = (1, 0)$ and $x_0 = (-1, 0)$, and let A be a small arc centered at a_0. If $f:(X, A) \to (X, A)$ is a map homotopic to the identity of (X, A), has the fixed point set $\{a_0, x_0\}$ and moves all points $x \in X - \{a_0, x_0\}$ towards a_0 along the arc from x_0 through x to a_0, then f has no fixed points on Bd A. But $\tilde{n}(f; X, A) = \tilde{N}(f; X, A) = 1$.

Note also that $N(f; X, A) = 1$, and that f has 2 fixed points. Higher-dimensional examples can easily be obtained by using a "fat" torus $S^1 \times B^n$, where B^n is an n-ball, and composing f with a retraction of $S^1 \times B^n$ onto S^1.

The map f in Example 3.3 which is fixed point free on Bd A has more than $N(f; X, A)$ fixed points on X. We now proceed to show that this behaviour is typical, and that maps of (X, A) with minimal fixed point sets must in fact have at least $\tilde{n}(f; X, A)$ fixed points on Bd A. Thus $\tilde{n}(f; X, A)$ does serve as a lower bound for the number of fixed points on Bd A, but only for the class of selfmaps of (X, A) which have a minimal fixed point set. To obtain this result, we need a formula which relates $\tilde{N}(f; X, A)$ and $\tilde{n}(f; X, A)$ and is similar to the definition (2.1) of $N(f; X, A)$.

Theorem 3.4. *If $f:(X,A)\to(X,A)$ is a map in $\mathscr{F}$, then $\tilde{N}(f;X,A)=\tilde{n}(f;X,A)+N(f)-N(f,\bar{f})$ and hence $N(f;X,A)=N(\bar{f})+\tilde{N}(f;X,A)-\tilde{n}(f;X,A)$.*

Proof. If we denote the fixed point classes of $f:X\to X$ by $\mathbb{F}$, then by definition

$$\begin{aligned}&\tilde{n}(f;X,A)+[N(f)-N(f,\bar{f})]\\&\quad=\#\{\mathbb{F}\mid\mathbb{F}\text{ is a common fixed point class of } f \text{ and } \bar{f} \text{ and does not assume its index in } A\}\\&\qquad+\#\{\mathbb{F}\mid\mathbb{F}\text{ is an essential fixed point class which is not a common fixed point class of } f \text{ and } \bar{f}\}.\end{aligned}$$

But a fixed point class $\mathbb{F}$ which is not a common fixed point class of f and $\bar{f}$ is essential if and only if it does not assume its index in A, and so we get

$$\begin{aligned}&\tilde{n}(f;X,A)+[N(f)-N(f,\bar{f})]\\&\quad=\#\{\mathbb{F}\mid\mathbb{F}\text{ is a common fixed point class of } f \text{ and } \bar{f} \text{ and does not assume its index in } A\}\\&\qquad+\#\{\mathbb{F}\mid\mathbb{F}\text{ is not a common fixed point class of } f \text{ and } \bar{f} \text{ and does not assume its index in } A\}\\&\quad=\#\{\mathbb{F}\mid\mathbb{F}\text{ does not assume its index in } A\}=\tilde{N}(f;X,A).\end{aligned}$$

Hence Theorem 3.4 is true. □

The lower bound property of $\tilde{n}(f;X,A)$ follows now immediately from Theorems 3.1 and 3.4 and the fact that $\bar{f}:A\to A$ must have at least $N(\bar{f})$ fixed points.

Theorem 3.5 (Lower bound). *If $f:(X,A)\to(X,A)$ is a map in $\mathscr{F}$ which has $N(f;X,A)$ fixed points, then it has at least $\tilde{n}(f;X,A)$ fixed points on* Bd A.

Although the computation of $\tilde{N}(f;X,A)$ and $\tilde{n}(f;X,A)$ requires the knowledge of all fixed point classes of $f:X\to X$ and $\bar{f}:A\to A$, it is easy in some special cases, and often possible in concrete examples, once $N(f)$ and $N(\bar{f})$ are known. The next two theorems, as well as the two examples following them, illustrate this fact.

Theorem 3.6. *If $f:(X,A)\to(X,A)$ is a map in $\mathscr{F}$ and if $N(\bar{f})=0$, then $\tilde{n}(f;X,A)=0$ and $\tilde{N}(f;X,A)=N(f)$.*

Proof. If $N(\bar{f})=0$, then there cannot exist any common fixed point classes of f and $\bar{f}$, so $N(f,\bar{f})=\tilde{n}(f;X,A)=0$. Hence $\tilde{N}(f;X,A)=N(f)$ by Theorem 3.4. □

Theorem 3.7. *Let $f:(X,A)\to(X,A)$ be a map in $\mathscr{F}$. If either X is simply connected or if X is connected and f is homotopic to the identity map* $\mathrm{id}:(X,A)\to(X,A)$, *then*

$$\text{(i)}\qquad \tilde{N}(f;X,A)=\begin{cases}1 & \text{if } L(f)\neq L(\bar{f}),\\ 0 & \text{if } L(f)=L(\bar{f}),\end{cases}$$

$$\text{(ii)}\qquad \tilde{n}(f;X,A)=\begin{cases}1 & \text{if } L(f)\neq L(\bar{f}) \text{ and } N(\bar{f})\neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. If $f:(X, A)\to(X, A)$ is as assumed in the theorem, then either $\operatorname{Fix} f=\emptyset$ or $f: X\to X$ has one fixed point class $\mathbb{F}_1$ with $\operatorname{ind}(X, f, \mathbb{F}_1)=L(f)$.

(i) In the case $\operatorname{Fix} f=\emptyset$ we have $\operatorname{Fix}\bar{f}=\emptyset$, and hence $L(f)=L(\bar{f})=0$ and $\tilde{N}(f; X, A)=0$. If, on the on the other hand, $\operatorname{Fix} f\neq\emptyset$, then $\operatorname{Fix}\bar{f}\subset\operatorname{Fix} f$ implies

$$\operatorname{ind}(A, \bar{f}, \mathbb{F}_1\cap A)=\operatorname{ind}(A, \bar{f}, \operatorname{Fix}\bar{f})=L(\bar{f}),$$

and so (i) follows from the definition of $\tilde{N}(f; X, A)$.

(ii) If $L(f)\neq L(\bar{f})$ and $N(\bar{f})\neq 0$, then $\operatorname{Fix} f\neq\emptyset$, and the one fixed point class $\mathbb{F}_1$ of $f: X\to X$ is a common fixed point class of f and $\bar{f}$ (as $N(\bar{f})\neq 0$) and does not assume its index in A (as $L(f)\neq L(\bar{f})$). Thus $\tilde{n}(f; X, A)=1$. If $L(f)=L(\bar{f})$, then $\tilde{n}(f; X, A)\leq\tilde{N}(f; X, A)$ implies $\tilde{n}(f; X, A)=0$, and if $N(\bar{f})=0$, then $\tilde{n}(f; X, A)=0$ by its definition. □

Example 3.8. Let X be a metrizable AR, let $A=\bigcup A_j$ be the disjoint union of finitely many AR's and let $f:(X, A)\to(X, A)$ be a map in $\mathscr{F}$. If there are k components A_j of A with $f(A_j)\subset A_j$, then it follows from Theorem 3.7 that

$$\tilde{N}(f; X, A)=\begin{cases}1 & \text{if } k\neq 1,\\ 0 & \text{if } k=1,\end{cases}$$

and

$$\tilde{n}(f; X, A)=\begin{cases}1 & \text{if } k\geq 2,\\ 0 & \text{if } k=0, 1.\end{cases}$$

(This example is modelled on [1, Theorem 5.1] where it is shown that if $f:(X, A)\to(X, A)$ is a compact map and all AR's A_j are open or closed in X, then $k\neq 1$ implies that f has a fixed point on $\operatorname{Cl}(X-A)$.)

The next example may seem somewhat contrived, but was chosen to illustrate the influence of inessential fixed point classes on the set of fixed points on $\operatorname{Cl}(X-A)$.

Example 3.9. Let X be the "fat" torus in Euclidean space $\mathbb{R}^3$ obtained by rotating the annulus $\frac{1}{4}\leq(x_1-2)^2+x_3^2\leq 1$ in the x_1x_3-plane around the x_3-axis, and let A be the disjoint union of n small 3-balls B_j^3 contained in the interior of X, and positioned so that $f(B_j^3)\subset B_j^3$ for all B_j^3 if $f:(X, A)\to(X, A)$ is the map given by $f(x_1, x_2, x_3)=(-x_1, x_2, x_3)$. (See Fig. 1, which shows the intersection of (X, A) with the x_2x_3-plane.) Then $N(f)=0$ [6, Example 2, p. 33 and Theorem 5.4, p. 21], but as $\operatorname{Fix} f$ is the intersection of X with the x_2x_3-plane, it is not empty. It consists of two annuli, each of which forms an inessential fixed point class $\mathbb{F}_1$ resp. $\mathbb{F}_2$ of $f: X\to X$. The fixed point set of $\bar{f}: A\to A$ consists of $n=k_1+k_2$ disks, where k_i, for $i=1, 2$, is the number of components of A which intersect $\mathbb{F}_i$. Hence

$$\operatorname{ind}(X, f, \mathbb{F}_i)=0,$$

but

$$\operatorname{ind}(A, \bar{f}, \mathbb{F}_i\cap A)=k_i \quad \text{for } i=1, 2.$$

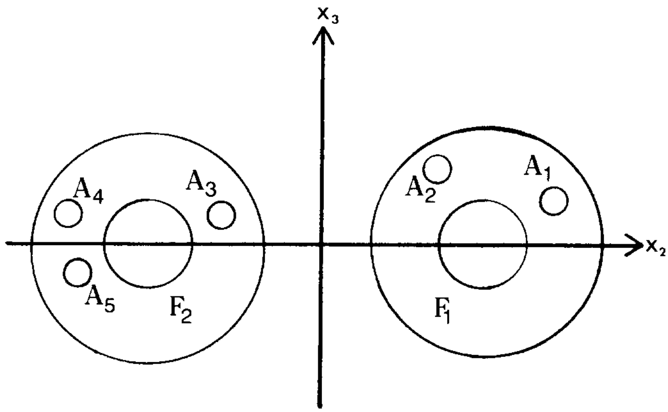

Fig. 1. $k_1 = 2$, $k_2 = 3$.

So we obtain (e.g. from the definition of $\tilde{N}(f; X, A)$ and Theorem 3.4)

$$\tilde{N}(f; X, A) = \tilde{n}(f; X, A) = \begin{cases} 0 & \text{if } k_1 = k_2 = 0, \\ 1 & \text{if } k_1 > 0, k_2 = 0 \text{ or } k_1 = 0, k_2 > 0, \\ 2 & \text{if } k_1 > 0 \text{ and } k_2 > 0. \end{cases}$$

We shall return to these two examples at the end of Section 5.

4. Some properties of the Nielsen numbers of the complement and of the boundary

The relative as well as the ordinary Nielsen numbers are invariant under homotopy and homotopy type. The same is true for $\tilde{N}(f; X, A)$ and $\tilde{n}(f; X, A)$.

Theorem 4.1 (Homotopy invariance). *If $f, g : (X, A) \to (X, A)$ are $\mathscr{F}$-homotopic maps in $\mathscr{F}$, then $\tilde{N}(f; X, A) = \tilde{N}(g; X, A)$ and $\tilde{n}(f; X, A) = \tilde{n}(g; X, A)$.*

Proof. Let $H : (X \times I, A \times I) \to (X, A)$ be an $\mathscr{F}$-homotopy from f to g, and let $\mathbb{F}$ be a fixed point class of $f : X \to X$ with $\operatorname{ind}(X, f, \mathbb{F}) \neq \operatorname{ind}(A, \bar{f}, \mathbb{F} \cap A)$. We consider two cases.

(i) If $\mathbb{F}$ is a common fixed point class of f and $\bar{f}$, then

$$\mathbb{F} \cap A = \bar{\mathbb{F}}_1 \cup \bar{\mathbb{F}}_2 \cup \cdots \cup \bar{\mathbb{F}}_k \cup \bar{F}',$$

where each $\bar{\mathbb{F}}_j$ (for $j = 1, 2, \ldots, k$ and $k \geq 1$) is an essential fixed point class of $\bar{f} : A \to A$ and $\bar{F}'$ is either empty or the union of inessential fixed point classes of $\bar{f} : A \to A$. Let $\bar{\mathbb{G}}_j$ be the essential fixed point class of $\bar{g} : A \to A$ which is related to $\bar{\mathbb{F}}_j$ by the homotopy $\bar{H} : A \times I \to A$. (See e.g. [2, Ch. IV, D and E, p. 87 ff.] [6, Ch. I.2,

p. 7 ff.] and [9, pp. 83–84] for the relation of essential fixed point classes under homotopies.) If $\mathbb{G}_j$ is the fixed point class of $g: X \to X$ which contains $\bar{\mathbb{G}}_j$, then it follows as in the proof of [7, Theorem 3.3] that $\mathbb{F}$ is H-related to $\mathbb{G}_j$, and hence $\mathbb{G}_1 = \mathbb{G}_2 = \cdots = \mathbb{G}_k = \mathbb{G}$ is the fixed point class of $g: X \to X$ which is H-related to $\mathbb{F}$. Thus

$$\mathbb{G} \cap A = \bar{\mathbb{G}}_1 \cup \bar{\mathbb{G}}_2 \cup \cdots \cup \bar{\mathbb{G}}_k \cup \bar{G}',$$

where $\operatorname{ind}(A, \bar{f}, \bar{\mathbb{F}}_j) = \operatorname{ind}(A, \bar{g}, \bar{\mathbb{G}}_j)$ for $j = 1, 2, \ldots, k$. Using the homotopy H^{-1} from g to f we see that $\bar{G}'$ cannot contain an essential fixed point class of $\bar{g}: A \to A$, so $\operatorname{ind}(A, \bar{g}, \bar{G}') = 0$. As $\operatorname{ind}(X, f, \mathbb{F}) = \operatorname{ind}(X, g, \mathbb{G})$ it follows that H relates a fixed point class $\mathbb{F}$ of $f: X \to X$ with $\operatorname{ind}(X, f, \mathbb{F}) \neq \operatorname{ind}(A, \bar{f}, \mathbb{F} \cap A)$ which is a common fixed point class of f and $\bar{f}$, to a fixed point class $\mathbb{G}$ of $g: X \to X$ with $\operatorname{ind}(X, g, \mathbb{G}) \neq \operatorname{ind}(A, \bar{g}, \mathbb{G} \cap A)$ so that $\mathbb{G}$ is a common fixed point class of g and $\bar{g}$.

(ii) If $\mathbb{F}$ is not a common fixed point class of f and $\bar{f}$, then $\operatorname{ind}(A, \bar{f}, \mathbb{F} \cap A) = 0$ and hence $\operatorname{ind}(X, f, \mathbb{F}) \neq 0$, and so there exists a unique essential fixed point class $\mathbb{G}$ of $g: X \to X$ which is H-related to $\mathbb{F}$. This fixed point class $\mathbb{G}$ is not a common fixed point class of g and $\bar{g}$, as otherwise H^{-1} would (as in part (i)) relate $\mathbb{G}$ to a common fixed point class of f and $\bar{f}$. Hence H relates a fixed point class $\mathbb{F}$ of $f: X \to X$ which does not assume its index in A and is not a common fixed point class of f and $\bar{f}$ to fixed point class $\mathbb{G}$ of $g: X \to X$ which does not assume its index in A and is not a common fixed point class of g and $\bar{g}$.

Using again the homotopy H^{-1} from g and f we see that the relations defined in the cases (i) and (ii) are one-to-one, and therefore Theorem 4.1 holds. □

The proof of the next theorem can easily be obtained along the lines of the proof of [6, Ch. I, Theorem 5.2, p. 20] and is omitted.

Theorem 4.2 (Commutativity). *If $f: (X, A) \to (Y, B)$ and $g: (Y, B) \to (X, A)$ are maps of pairs of spaces and if gf and fg are maps in $\mathcal{F}$, then*

$$\tilde{N}(gf; X, A) = \tilde{N}(fg; Y, B) \quad \textit{and} \quad \tilde{n}(gf; X, A) = \tilde{n}(fg; Y, B).$$

It is also a straightforward task to extend the definition of maps of pairs of spaces of the same homotopy type given in [7, Section 3] to that of maps in $\mathcal{F}$ of the same $\mathcal{F}$-homotopy type, and to show (as in [6, Ch. I, Theorem 5.4, p. 21]; see also [7, Theorem 3.5]) that $\tilde{N}(f; X, A)$ and $\tilde{n}(f; X, A)$ are $\mathcal{F}$-homotopy type invariant. We leave the details to the reader.

We now relate $\tilde{N}(f; X, A)$ to existing results concerning fixed points on $\mathrm{Cl}(X - A)$. It was observed by Bowszyc [1] that Lefschetz numbers can be used to obtain a sufficient condition for the existence of a fixed point on $\mathrm{Cl}(X - A)$. One of his main results, which was extended to maps of compact attraction in [3] and [4], is the following.

Theorem 4.3 [1, Theorems 4.4 and 4.5]. *Let (X, A) be a pair of metrizable ANR's, let A be either closed or open in A, and let $f:(X, A)\to(X, A)$ be a compact map. If $L(f)\neq L(\bar{f})$, then f has a fixed point on* $\mathrm{Cl}(X-A)$.

In the classical setting, where $A=\emptyset$, it follows from $L(f)\neq 0$ that $N(f)>0$. The next theorem shows how this fact generalizes to maps of pairs of spaces.

Theorem 4.4. *If $f:(X, A)\to(X, A)$ is a map in $\mathcal{F}$ and $L(f)\neq L(\bar{f})$, then $\tilde{N}(f; X, A)>0$.*

Proof. We will show that $\tilde{N}(f; X, A)=0$ implies $L(f)=L(\bar{f})$.

It follows from the normalization and additivity axioms of the index that

$$\begin{aligned} L(f) &= \mathrm{ind}(X, f, X) \\ &= \sum\{\mathrm{ind}(X, f, \mathbb{F})\,|\,\mathbb{F} \text{ is a fixed point class of } f:X\to X\} \end{aligned}$$

and

$$\begin{aligned} L(\bar{f}) &= \mathrm{ind}(A, \bar{f}, A) \\ &= \sum\{\mathrm{ind}(A, \bar{f}, \bar{\mathbb{F}})\,|\,\bar{\mathbb{F}} \text{ is a fixed point class of } \bar{f}:A\to A\}. \end{aligned}$$

But each fixed point class $\bar{\mathbb{F}}$ of $\bar{f}:A\to A$ is contained in exactly one fixed point class $\mathbb{F}$ of $f:X\to X$, and so

$$L(\bar{f})=\sum\{\mathrm{ind}(A, \bar{f}, \mathbb{F}\cap A\,|\,\mathbb{F} \text{ is a fixed point class of } f:X\to X\}.$$

By definition $\tilde{N}(f; X, A)=0$ implies that

$$\mathrm{ind}(X, f, \mathbb{F})=\mathrm{ind}(A, \bar{f}, \mathbb{F}\cap A)$$

for every fixed point class $\mathbb{F}$ of $f:X\to X$, and hence $L(f)=L(\bar{f})$.

5. The location of minimal fixed point sets

The Nielsen number $N(f)$ is not only a lower bound for the number of fixed points on X for all maps in the homotopy class of $f:X\to X$, but for a large class of compact polyhedra it is the sharp lower bound, i.e. there exists a map $g:X\to X$ homotopic to f which has precisely $N(f)$ fixed points. (See e.g. [5, Theorem 5.2].) For maps $f:(X, A)\to(X, A)$ of pairs of suitable compact polyhedra the sharp lower bound for the number of fixed points on X is the relative Nielsen number $N(f; X, A)$ [7, Theorem 6.2], but the location of the $N(f; X, A)$ fixed points of a map $g:(X, A)\to(X, A)$ homotopic to $f:(X, A)\to(X, A)$ is not arbitrary. As $\bar{g}:A\to A$ is homotopic to $\bar{f}:A\to A$, there must be at least $N(\bar{f})$ fixed points on A, and due to Theorems 3.1 and 3.5 there must be at least $\tilde{N}(f; X, A)$ fixed points on $\mathrm{Cl}(X-A)$, and at least $\tilde{n}(f; X, A)$ fixed points on $\mathrm{Bd}\,A$ if the fixed point set of g is minimal. We now obtain conditions for (X, A) so that a map $g:(X, A)\to(X, A)$ which is homotopic to $f:(X, A)\to(X, A)$ and has a minimal fixed point set can be constructed in such a way that $N(\bar{f})$ of these fixed points lie on A, $\tilde{N}(f; X, A)$ lie on $\mathrm{Cl}(X-A)$ and $\tilde{n}(f; X, A)$ lie on $\mathrm{Bd}\, A$. Hence we show that for such pairs $N(\bar{f})$ and $\tilde{N}(f; X, A)$ are sharp lower bounds for the number of fixed points on A and on $\mathrm{Cl}(X-A)$ for all maps homotopic to $f:(X, A)\to(X, A)$, and that $\tilde{n}(f; X, A)$ is a sharp lower bound for the number of fixed points on $\mathrm{Bd}\, A$ for all maps homotopic to $f:(X, A)\to(X, A)$ which have a minimal fixed point set. The conditions for (X, A) consist of those of [7, Theorem 6.2] and one further assumption: namely, that each component of A has a non-empty interior. This assumption is essential, as e.g. $\mathrm{Int}\, A=\emptyset$ implies $\mathrm{Cl}(X-A)=X$ and $\mathrm{Bd}\, A=A$, and the least number of fixed points on $\mathrm{Cl}(X-A)$ is then $N(f; X, A)$ and not $\tilde{N}(f; X, A)$ and the least number of fixed points on $\mathrm{Bd}\, A$ is then $N(\bar{f})$. If only some, but not all components of A have a non-empty interior, then the calculation of the minimum number of fixed points on $\mathrm{Cl}(X-A)$ and $\mathrm{Bd}\, A$ is likely to be messy. As in [7] we say that a subspace A of X *can be by-passed* if every path in X with end points in $X-A$ is homotopic, keeping end points fixed, to a path in $X-A$.

Theorem 5.1 (Minimum Theorem). *Let X, A be a pair of compact polyhedra such that*

(i) *X is connected,*

(ii) *$X-A$ has no local cut point and is not a 2-manifold,*

(iii) *every component of A is a Nielsen space with a non-empty interior,*

(iv) *A can be by-passed.*

Then every map $f:(X, A)\to(X, A)$ is homotopic to a map $g:(X, A)\to(X, A)$ which has $N(f; X, A)$ fixed points. Of these, $N(\bar{f})$ lie on A, $\tilde{N}(f; X, A)$ lie on $\mathrm{Cl}(X-A)$ *and $\tilde{n}(f; X, A)$ lie on* $\mathrm{Bd}\, A$.

Proof. Without loss of generality we can assume that $X\neq A$, as otherwise Theorem 5.1 is known. We proceed in three steps.

Step 1. We show that $f:(X, A)\to(X, A)$ is homotopic to a map $h:(X, A)\to(X, A)$ which has the following properties:

(a) $\bar{h}$ has $N(\bar{f})$ fixed points on A, of which $\tilde{n}(f; X, A)$ lie on $\mathrm{Bd}\, A$ and $N(\bar{f})-\tilde{n}(f; X, A)$ in $\mathrm{Int}\, A$,

(b) distinct fixed points of $\bar{h}$ on $\mathrm{Bd}\, A$ lie in distinct fixed point classes of $h: X\to X$, and these fixed point classes do not assume their index in A,

(c) there exists a compact polyhedron B in X so that $A\subset X-B$ and h is fixed point free on $\mathrm{Cl}(X-B)-A$.

To construct h we first homotope $\bar{f}: A\to A$ to a map $\bar{f}': A\to A$ with $N(\bar{f})$ fixed points a_j which all lie in maximal simplexes of A, write $\bar{H}$ for the homotopy from $\bar{f}$ to $\bar{f}'$, and label the essential fixed point classes $\bar{\mathbb{F}}_j$ of $\bar{f}: A\to A$ so that $\bar{\mathbb{F}}_j$ is $\bar{H}$-related to the essential fixed point class $\bar{\mathbb{F}}_j'=\{a_j\}$ of $\bar{f}'$. For each common fixed point class $\mathbb{F}$ of f and $\bar{f}$ which does not assume its index in A we select exactly one label j so that $\bar{\mathbb{F}}_j\subset\mathbb{F}$. Then we use the fact that A is a Nielsen space to move all a_j whose label was selected to Bd A, and keep all other a_j in Int A. Let the resulting map, which is homotopic to $\bar{f}: A \to A$, be $\bar{h}: A \to A$. If $\bar{h}: A \to A$ is extended to a map $h: (X, A) \to (X, A)$ as in Step 1 of the proof of [7, Theorem 4.1] then h satisfies (a), (b) and (c).

Step 2. We show that $f: (X, A) \to (X, A)$ is homotopic to a map $h': (X, A) \to (X, A)$ with the following properties:

(a) $\bar{h}'$ has $N(\bar{f})$ fixed points on A, of which $\tilde{n}(f; X, A)$ lie on Bd A and $N(\bar{f}) - \tilde{n}(f; X, A)$ in Int A,

(b) distinct fixed points of $\bar{h}'$ on Bd A lie in distinct fixed point classes of $h': X \to X$, and these fixed point classes do not assume their index in A,

(c) h' is fix-finite, and all fixed points on $X - A$ lie in maximal simplexes.

The construction of h' from h is the same as the construction of g from h in Step 2 of the proof of [7, Theorem 4.1].

Step 3. We show that $f: (X, A) \to (X, A)$ is homotopic to a map $g: (X, A) \to (X, A)$ which has $N(f; X, A)$ fixed points, of which $N(\bar{f})$ lie on A, $\tilde{N}(f; X, A)$ lie on $\mathrm{Cl}(X - A)$ and $\tilde{n}(f; X, A)$ on Bd A.

As in the proof of [7, (6.3)] we can unite all fixed points of h' which lie in $X - A$ and belong to the same fixed point class of $h': X \to X$ to a single fixed point in $X - A$, and thus obtain a map $g': (X, A) \to (X, A)$ so that each fixed point class of $g': X \to X$ contains at most one point on $X - A$ and at most finitely many points on A, and so that (a) and (b) of Step 2 (with g' instead of h') are satisfied. We now deal with the fixed point classes $\mathbb{G}'$ of $g': X \to X$ according to the following four cases.

(i) If $\mathbb{G}'$ assumes its index in A and is a common fixed point class of g' and $\bar{g}'$, then either $\mathbb{G}' \subset A$ or $\mathbb{G}' = \{x_0\} \cup \bar{G}'_0$ where $x_0 \in X - A$ and $\bar{G}'_0$ consists of one or more points in Int A. As $\mathbb{G}'$ assumes its index in A, we have

$$\begin{aligned}\mathrm{ind}(A, \bar{g}', \mathbb{G}' \cap A) &= \mathrm{ind}(X, g', \mathbb{G}') \\ &= \mathrm{ind}(X, g', x_0) + \mathrm{ind}(X, g', \bar{G}'_0) \\ &= \mathrm{ind}(X, g', x_0) + \mathrm{ind}(A, \bar{g}', \bar{G}'_0) \\ &= \mathrm{ind}(X, g', x_0) + \mathrm{ind}(A, \bar{g}', \mathbb{G}' \cap A).\end{aligned}$$

Hence $\mathrm{ind}(X, g', x_0) = 0$, and we can delete x_0 in the usual manner [2, Ch. VIII B, Theorem 4, p. 123].

(ii) If $\mathbb{G}'$ assumes its index in A and is not a common fixed point class of g' and $\bar{g}'$, then $\mathbb{G}'$ must consist of a single point $x_0 \in X - A$ with $\mathrm{ind}(X, g', x_0) = 0$, and we can again delete x_0.

(iii) If $\mathbb{G}'$ does not assume its index in A and is a common fixed point class of g' and $\bar{g}'$, then there exists according to (b) of Step 2 an isolated fixed point $a_j \in \mathbb{G}' \cap \mathrm{Bd}\, A$, and we can use the method of the proof of [7, (6.3)] to remove the single fixed point of $\mathbb{G}'$ on $X - A$ (if it exists) by uniting it with $a_j \in \mathrm{Bd}\, A$.

(iv) If, finally, $\mathbb{G}'$ does not assume its index in A and is not a common fixed point class of g' and $\bar{g}'$, then

$$\mathrm{ind}(X, g', \mathbb{G}') \neq \mathrm{ind}(A, \bar{g}', \mathbb{G}' \cap A) = 0,$$

so $\mathbb{G}'$ must be essential. Hence $\mathbb{G}' = \{x_0\} \in X - A$ with $\operatorname{ind}(X, g', x_0) \neq 0$, and therefore x_0 cannot be deleted. Now a fixed point class $\mathbb{G}'$ which is not a common fixed point class of g' and $\bar{g}'$ is in fact essential if and only if it does not assume its index in A. As there are $N(g') - N(g', \bar{g}') = N(f) - N(f, \bar{f})$ essential fixed point classes of $g' : X \to X$ which are not common fixed point classes of g' and $\bar{g}'$, we are left with $N(f) - N(f, \bar{f})$ fixed points on $X - A$.

Let $g : (X, A) \to (X, A)$ be the map obtained from $g' : (X, A) \to (X, A)$ after all fixed point classes have been dealt with according to the four cases. We now count the number of fixed points of g. There are by construction $N(\bar{f})$ fixed points on A, of which $\tilde{n}(f; X, A)$ lie on Bd A, and there are $N(f) - N(f; \bar{f})$ fixed points on $X - A$. Hence it follows from Theorem 3.4 that the number of fixed points on $\operatorname{Cl}(X - A) = \operatorname{Bd} A \cup \operatorname{Int}(X - A)$ is

$$\tilde{n}(f; X, A) + N(f) - N(f, \bar{f}) = \tilde{N}(f; X, A),$$

and on $X = \operatorname{Cl}(X - A) \cup \operatorname{Int} A$ it is (again by Theorem 3.4)

$$\tilde{N}(f; X, A) + N(\bar{f}) - \tilde{n}(f; X, A) = N(f; X, A).$$

Thus Theorem 5.1 holds. □

Note that Example 3.9 was constructed so that the assumptions of Theorem 5.1 are satisfied, and therefore the Nielsen numbers of the complement and of the boundary calculated there can actually be realized by a map with a minimal fixed point set. Another example where Theorem 5.1 applies can be obtained as a special case of Example 3.8 if X is a closed n-ball, with $n \geq 3$, and all A_j are closed n-balls contained in Int X.

Remark 5.2. If $f : (X, A) \to (X, A)$ is homotopic to the identity $\operatorname{id} : (X, A) \to (X, A)$, then all Nielsen numbers occuring in Theorem 5.1 can be computed, and the location of a minimal fixed point set can be determined, in terms of the Euler characteristics of X and the components of A. This was done in [8, Theorem 4.1], where it was also shown that in this case the assumptions on (X, A) can be relaxed.

Remark 5.3. Theorem 5.1 permits us to homotope a map to one with a minimal fixed point set in which the fixed points are distributed so that there are $N(\bar{f})$ of them on A, $\tilde{N}(f; X, A)$ on $\operatorname{Cl}(X - A)$, and $\tilde{n}(f; X, A)$ on Bd A. This does not imply, however, that every selfmap of (X, A) with a minimal fixed point set *must* have the fixed points distributed in this manner, even if the assumptions of Theorem 5.1 are satisfied. If, e.g. $X = B^4$ is the unit ball $\{x \mid \|x\| \leq 1\}$ in $\mathbb{R}^4$, if $A = \{x \mid \frac{1}{2} \leq \|x\| \leq 1\}$ and if $f : (X, A) \to (X, A)$ is the identity, then $N(\bar{f}) = 0$, $\tilde{N}(f; X, A) = 1$ and $\tilde{n}(f; X, A) = 0$. Theorem 4.1 (ii) of [8] shows that there exists a deformation g of (X, A) which has $N(f; X, A)$ fixed points on Bd A and no further fixed points. Hence g has $1 > N(\bar{f})$ fixed point on A and $1 > \tilde{n}(f; X, A)$ fixed point on Bd A. It is also easy to construct a map with a minimal fixed point set and greater than $\tilde{N}(f; X, A)$ fixed points on $\mathrm{Cl}(X-A)$. In fact, the proof of Theorem 6.2 in [7] shows that all $N(\bar{f})$ fixed points can be moved onto Bd A, and hence all $N(f; X, A)$ fixed points can lie on $\mathrm{Cl}(X-A)$. A concrete example of a map with a minimal fixed point set but greater than $\tilde{N}(f; X, A)$ fixed points on $\mathrm{Cl}(X-A)$ can be obtained with $X=B^3$, $A=\{x \mid 0 \leq \|x\| \leq \frac{1}{2}\}$ by taking $f:(X, A) \to (X, A)$ as the identity map and using [8, Theorem 4.1 (iii)] to show that there exists a deformation g of (X, A) with exactly $N(f; X, A)=1$ fixed point a_1 so that a_1 lies on Bd A. Hence g has $1 > \tilde{N}(f; X, A)$ fixed point on $\mathrm{Cl}(X, A)$.

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References

[1] C. Bowszyc, Fixed point theorems for the pairs of spaces, Bull. Acad. Polon. Sci. Sér. Sci., Math., Astr. et Phys. 16 (1968) 845–850.
[2] R.F. Brown, The Lefschetz Fixed Point Theorem (Scott, Foresman, Glenview, IL, 1971).
[3] Z. Dzedzej, Fixed point index theory for a class of non-acyclic maps, Dissertationes Math. 253 (1985) 58.
[4] L. Górniewicz and A. Granas, On a theorem of C. Bowszyc concerning the relative version of the Lefschetz fixed point theorem, Bull. Inst. Math. Acad. Sinica 13 (1985) 137–142.
[5] Boju Jiang, On the least number of fixed points, Amer. J. Math. 102 (1980) 749–763.
[6] Boju Jiang, Lectures on Nielsen Fixed Point Theory, Contemporary Mathematics 14 (Amer. Math. Society, Providence).
[7] H. Schirmer, A relative Nielsen number, Pacific J. Math. 122 (1986) 459–473.
[8] H. Schirmer, Fixed point sets of deformations of pairs of spaces, Topology Appl. 23 (1986) 193–205.
[9] K. Scholz, The Nielsen fixed point theory for noncompact spaces, Rocky Mountain J. Math. 4 (1974) 81–87.